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Regular over-orders of lattice-finite rings and the Krull–Schmidt property

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Abstract

Let R be a lattice-finite noetherian semilocal ring without simple left ideals. In Rump (Preprint) we prove that R is an order in a semisimple ring Q. We refine this result by showing that R has a semiperfect regular over-order if the category R-**lat** of R-lattices has the Krull–Schmidt property. Together with the results of Rump this implies that both conditions are equivalent to the existence of almost split sequences in R-**lat**.

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1. Introduction

A left artinian ring R is said to be *representation-finite* if there are only finitely many isomorphism classes of indecomposable finitely generated left R-modules [3,1]. Such a ring R is also right artinian, and the category R-**mod** of finitely generated left R-modules is a *strict* τ*-category* [4], that is, a Krull–Schmidt category with left and right almost split sequences for all of its objects (see Section 2 for precise definitions). By [9], we have the following one-dimensional analogue of these results.

Let R be a left noetherian semilocal ring. Define an R*-lattice* as a finitely generated left R-module without simple submodules. We call R *lattice-finite* if there are only finitely many isomorphism classes of indecomposable R-lattices. For the study of R-**lat**, the category of R-lattices, we may assume that the regular representation ${}_RR$ is an R-lattice. (Otherwise, the largest length-finite left ideal of R is two sided, and R-**lat** =

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R/I-**lat**.) In [9] we show that in case R is lattice finite with ${}_RR \in R$-**lat**, the maximal left quotient ring Q of R is semisimple. If, in addition, R is (two sided) noetherian, then R is an order in Q [9, Theorem 1]. However, in contrast to the artinian situation, lattice-finiteness of R does not imply that R-**lat** has almost split sequences. For example, the local ring R of the plane curve $C: y^2 = x^3 + x^2$ at the singularity (0,0) is lattice-finite. If $\mathbb{A}^1 \twoheadrightarrow C$ is the normalization of C, then the indecomposable R-lattices are ${}_RR$ and ${}_RR'$, where R' is the localization of $\mathbb{R}[\mathbb{A}^1]$ at the two points of the affine line $\mathbb{A}^1$ lying over the singularity of C. Hence, $\mathrm{End}_R(R')$ is not local, and so there cannot exist a left or right almost split sequence for ${}_RR'$ in R-**lat**. Thus, in the commutative case, the existence of almost split sequences in R-**lat** requires that the scheme R has a semiperfect normalization.

More generally, we prove in [9, Theorem 2], that for a lattice-finite ring R, the category R-**lat** is a strict τ-category if R has a semiperfect *regular* left over-order, that is, a left over-order of the form $R' \cong \mathrm{M}_{n_1}(\Omega_1) \times \cdots \times \mathrm{M}_{n_s}(\Omega_s)$, where Ω_i are (non-commutative) discrete valuation domains. Here, we call R' a *left over-order* of R if R is a subring of R' with ${}_RR' \in R$-**lat** such that the left conductor $\{a \in R \mid R'a \subset R\}$ of R' in R has finite index in ${}_RR'$.

In this note, we prove a converse when R is noetherian. Assume that R is a lattice-finite noetherian semilocal ring with ${}_RR \in R$-**lat**. We show that whenever R-**lat** is a Krull–Schmidt category, R has a two-sided semiperfect regular over-order. By [9] this shows that the existence of a semiperfect regular over-order of R is equivalent to the Krull–Schmidt property of R-**lat** and implies, in addition, that R-**lat** has almost split sequences.

For the theory of classical orders and their lattices we refer the reader to [7]. Standard concepts like Auslander–Reiten sequences are treated, e.g., in [1,2].

2. Prerequisites

Let R be a left noetherian semilocal ring. By R-**mod** we denote the category of finitely generated left R-modules, and $l_R(M)$ will be the length of an R-module M. We call $E \in R$-**mod** an *R-lattice* [9] if E has no simple submodules. The full subcategory of R-lattices in R-**mod** will be denoted by R-**lat**. If E is a submodule of $E' \in R$-**lat**, then E' is said to be an *over-lattice* of E if E'/E is of finite length. We call R *lattice-finite* if there are only finitely many isomorphism classes of indecomposable R-lattices. If R is lattice-finite, then [9], Proposition 2, implies that the Jacobson radical

$$\mathrm{Rad}: R\text{-}\mathbf{lat} \to R\text{-}\mathbf{lat} \qquad (1)$$

has a right adjoint $\mathrm{Rad}^\circ : R$-**lat** $\to R$-**lat** which associates a proper over-lattice $\mathrm{Rad}^\circ E \supsetneq E$ to every R-lattice $E \neq 0$. We call $\mathrm{Rad}^\circ E$ the *upper radical* of E. Another property of lattice-finite rings follows by [9], Corollary of Proposition 3.

Proposition 1. *Let R be a lattice-finite left noetherian semilocal ring. If $E \in R$-**lat** is an essential submodule of $F \in R$-**mod**, then F is an over-lattice of E.*

For the study of R-**lat** we may assume without loss of generality that $_RR \in R$-**lat**. Then Proposition 1 implies that for the injective envelope Q of $_RR$, every R-linear map $_RR \to Q$ has a unique extension to Q. Therefore, Q can be identified with $\mathrm{End}_R(Q)^{\mathrm{op}}$ and becomes a semisimple ring with R as a subring. By [6], Q coincides with the maximal left quotient ring of R in the sense of Utumi. Moreover, the injective envelope of any $E \in R$-**lat** is $Q \otimes_R E$ (see [9, Theorem 1]).

Recall that a subring R of a ring Q is said to be a *left* (*right*) *order* in Q if the regular elements of R are invertible in Q, and every element of Q can be written in the form $r^{-1}a$ (resp. ar^{-1}) with $a, r \in R$ and r regular. A left and right order in Q is said to be an *order* in Q.

We call a full subcategory $\mathscr{B}$ of an additive category $\mathscr{A}$ *strictly rejective* (cf. [5], 5.1) if $\mathscr{B}$ is closed with respect to isomorphisms, and for each object $A \in \mathscr{A}$ there are *regular* (i.e. monic and epic) morphisms $A_0 \xrightarrow{r} A \xrightarrow{s} A^\circ$ in $\mathscr{A}$ with $A_0, A^\circ \in \mathscr{B}$ such that every morphism $B \to A$ (resp. $A \to B$) with $B \in \mathscr{B}$ factors through r (resp. s). An additive category $\mathscr{A}$ is said to be a *Krull–Schmidt category* if every object of $\mathscr{A}$ has a finite decomposition into objects with local endomorphism rings. Then the ideal $\mathrm{Rad}\,\mathscr{A}$ generated by the non-invertible morphisms between indecomposable objects is said to be the *radical* of $\mathscr{A}$. A morphism $f: A \to B$ in $\mathscr{A}$ is said to be *right (left) almost split* if $f \in \mathrm{Rad}\,\mathscr{A}$, and every $C \to B$ (resp. $A \to C$) in $\mathrm{Rad}\,\mathscr{A}$ factors through f. A sequence

$$\tau A \overset{v}{\rightarrowtail} \vartheta A \xrightarrow{u} A \tag{2}$$

in $\mathscr{A}$ is called *right almost split* (for A) if u and v are right resp. left almost split morphisms, and $v = \ker u$. It is well known that up to isomorphism, a right almost split sequence (2) is uniquely determined by A. *Left almost split sequences*

$$A \rightarrowtail \vartheta^- A \twoheadrightarrow \tau^- A \tag{3}$$

for A are defined in a dual way. If left and right almost split sequences exist for every object $A \in \mathscr{A}$, then $\mathscr{A}$ is said to be a *strict τ-category* [4].

Now let R be a subring of a ring R'. We call R' a *left over-order* of R if $_RR' \in R$-**lat**, and $(R:R')_r := \{a \in R \mid R'a \subset R\}$ has finite index in $_RR'$. We make no difference between two left over-orders R' and R'' if there is a ring isomorphism $e: R' \xrightarrow{\sim} R''$ with $e|_R = 1_R$. By [9], Proposition 4, a left over-order R' of R is left noetherian (left noetherian semilocal; a left order in a semisimple ring Q) if and only if R is so. Furthermore, every left over-order R' of a left noetherian ring R with $\mathrm{Soc}_R R = 0$ gives rise to a full embedding

$$R'\text{-}\mathbf{lat} \hookrightarrow R\text{-}\mathbf{lat}, \tag{4}$$

which makes R'-**lat** into a strictly rejective full subcategory of R-**lat**, and this gives a one-to-one correspondence between left over-orders and strictly rejective full subcategories of R-**lat** ([9, Proposition 5]). In particular, this shows that the relation "left over-order" is transitive for left noetherian rings. *Right over-orders* of R are defined analogously, and a left and right over-order of R is called an *over-order* of R.

3. Regular over-orders

Let R be a left noetherian semilocal ring. In [8] we call R *regular* of dimension d if there are invertible ideals $P_1,\dots,P_n$ of R such that P_i is invertible modulo $P_1+\cdots+P_{i-1}$ for all i, and $P_1+\cdots+P_n=\operatorname{Rad} R$. Thus, R is regular of dimension one if and only if $\operatorname{Rad} R$ is invertible. If, in addition, R is semiperfect, then R is Morita equivalent to a product of rings of the form

$$\begin{pmatrix} \Omega\Pi & \cdots\cdots & \Pi \\ \vdots & \ddots & \vdots \\ \vdots & & \Pi \\ \Omega & \cdots\cdots & \Omega \end{pmatrix} \tag{5}$$

with a (non-commutative) discrete valuation domain Ω and $\Pi := \operatorname{Rad}\Omega$ (see [8, Proposition 1.6]). From now on, we will use the two-sided noetherian property.

Proposition 2. *Let R be a lattice-finite noetherian semilocal ring with ${}_RR\in R$-**lat**. Then there exists an over-order R_0 of R such that $\operatorname{Rad}^\circ \operatorname{Rad} R_0=R_0$ holds in R_0-**lat**.*

Proof. The adjunction $\operatorname{Rad}\dashv\operatorname{Rad}^\circ$ yields an isomorphism

$$\operatorname{Rad}^\circ \operatorname{Rad} R\cong \operatorname{Hom}_R({}_RR,\operatorname{Rad}^\circ \operatorname{Rad} R)\cong \operatorname{Hom}_R(\operatorname{Rad} R,\operatorname{Rad} R)$$

in R-**lat**. Hence, if we identify $\operatorname{Rad}^\circ\operatorname{Rad} R$ with $\operatorname{End}_R(\operatorname{Rad} R)^{\mathrm{op}}$, then the unit morphism $\eta_R: {}_RR\to\operatorname{Rad}^\circ \operatorname{Rad} R$ becomes a ring homomorphism. Since R is noetherian, we infer from [9, Theorem 1], that R is an order in a semisimple ring Q such that Q contains $R' := \operatorname{Rad}^\circ \operatorname{Rad} R$ as a subring. By [11, II, Theorem, 2.2], there is a regular element $r\in\operatorname{Rad} R$. Hence $rR'\subset R$ implies that $(R')_R\subset r^{-1}R\in R^{\mathrm{op}}$-**lat**. Since $l_R(R'/R)<\infty$, we get $R'(\operatorname{Rad} R)^n\subset R$ for some $n\in\mathbb{N}$. Consequently, R' is an over-order of R. So there is a full embedding (4) which preserves indecomposables. Hence R' is lattice-finite. Moreover, R' is noetherian and semilocal by [9, Proposition 4]. If $R'\neq R$, we can iterate the procedure, which yields the desired over-order R_0 after finitely many steps. □

Using this result, we get the following extension of the main theorem of [10].

Theorem. *Let R be a lattice-finite noetherian semilocal ring with ${}_RR\in R$-**lat**. The following are equivalent*:

(a) *R has a semiperfect one-dimensional regular left over-order.*
(b) *R has a semiperfect one-dimensional regular over-order.*
(c) *R-**lat** is a Krull–Schmidt category.*
(d) *R-**lat** is a strict τ-category.*

Proof. (a) ⇒ (d): Every regular ring (5) has an over-order of the form $\mathrm{M}_n(\Omega)$. Therefore, the assertion follows by [9, Theorem 2].

(c) $\Rightarrow$ (b): By Proposition 2, there is an over-order R_0 of R such that ${}_{R_0}R_0$ is the upper radical of Rad R_0. Since $R_0 \in R_0$-**lat** $\subset R$-**lat**, there is a decomposition $R_0 = P_1 \oplus \cdots \oplus P_n$ into indecomposable projective R_0-modules with $\mathrm{End}_{R_0}(P_i)$ local. By [9, Theorem 1], we know that R_0 is an order in a semisimple ring Q. Choose $i \in \{1, \ldots, n\}$ such that $l_Q(Q \otimes_{R_0} P_i)$ is maximal. Since $\mathrm{Rad}^\circ\, P_i \supsetneq P_i$, there exists an over-lattice P_i' of P_i with P_i'/P_i simple. Then $\mathrm{Rad}^\circ\, \mathrm{Rad}\, P_i = P_i$ implies that $\mathrm{Rad}\, P_i' = P_i$. Hence, for some $j \in \{1, \ldots, n\}$, there is an epimorphism $P_j \twoheadrightarrow P_i'/P_i$ which lifts to a homomorphism $p : P_j \to P_i'$ with $p(P_j) \not\subset P_i$. By Nakayama's lemma, this gives $p(P_j) = P_i'$. Since by assumption, $l_Q(Q \otimes_{R_0} P_j) \leqslant l_Q(Q \otimes_{R_0} P_i)$, we infer that p is an isomorphism. Therefore, $P_i' = \mathrm{Rad}^\circ\, P_i$. Applying the same argument to P_i' and continuing in this way, we get a composition series $P_i \subset P_i' \subset P_i'' \subset \cdots \subset P_i^{(m)} \cong P_i$ of R_0-lattices with $\mathrm{Rad}\, P_i^{(j+1)} = P_i^j$ and $\mathrm{Rad}^\circ\, P_i^j = P_i^{(j+1)}$ for $j < m$. Now let E be a non-zero R_0-sublattice of $Q \otimes_{R_0} P_i$. Since $Q \otimes_{R_0} P_i$ is the injective envelope of P_i (see [9, Theorem 1]), Proposition 1 implies that $E \subset (\mathrm{Rad}^\circ)^k P_i$ for some $k \in \mathbb{N}$. So we may assume that $E \subset P_i$. Since P_i is noetherian, there exists some $k \in \mathbb{N}$ with $(\mathrm{Rad}^\circ)^k E \subset P_i$ and $(\mathrm{Rad}^\circ)^k E \not\subset \mathrm{Rad}\, P_i$. By Nakayama's lemma, this implies that $(\mathrm{Rad}^\circ)^k E = P_i$. Hence $E = \mathrm{Rad}^k\, P_i$. Consequently, the R-lattices in $Q \otimes_{R_0} P_i$ form a chain. Therefore, $Q \otimes_{R_0} P_i$ is a simple Q-module, and $\mathrm{End}_{R_0}(P_i)$ is a discrete valuation domain. Now it follows that R_0 is regular of dimension one.

As the implications (d) $\Rightarrow$ (c) and (b) $\Rightarrow$ (a) are trivial, the proof is complete. $\square$

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